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Symmetry Classes of Tensors Associated with Certain Groups

RANDALL R. HOLMES and TIN-YAU TAM
Department of Algebra, Combinatorics and Analysis, Auburn University, AL 36849-5307

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We discuss the existence of an orthogonal basis consisting of decomposable vectors for some symmetry classes of tensors associated with certain subgroups of the full symmetric group. The dimensions of these symmetry classes of tensors are also given.

1. INTRODUCTION

Let V be a complex inner product space of dimension m. Let $\bigotimes^n V$ be the nth tensor power of V and write $v_1 \otimes \cdots \otimes v_n$ for the decomposable tensor product of the indicated vectors. To each permutation σ in the full symmetric group S_n there corresponds a linear operator $P(\sigma)$ determined by

$$P(\sigma)v_1 \otimes \cdots \otimes v_n = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}. \tag{1}$$

Let G be a subgroup of S_n. Let χ be an irreducible character of G. Define

$$T(G,\chi) = \frac{\chi(e)}{|G|} \sum_{\sigma \in G} \chi(\sigma)P(\sigma). \tag{2}$$

Let $\chi_1,\ldots,\chi_k$ be all the irreducible characters of G. It follows from the orthogonality relations for characters that $\{T(G,\chi_i) : i = 1,\ldots,k\}$ is a set of annihilating idempotents which sum to the identity, i.e.,

$$T(G,\chi_i)T(G,\chi_j) = T(G,\chi_i)\delta_{ij} \tag{3}$$

and

$$I_{\otimes^n V} = T(G,\chi_1) + \cdots + T(G,\chi_k). \tag{4}$$

Let χ be one of the χ_i's. The image of $\bigotimes^n V$ under the map $T(G,\chi)$ is called the *symmetry class of tensors associated with G and χ* and is denoted $V_\chi(G)$. The image of $v_1 \otimes \cdots \otimes v_n$ under $T(G,\chi)$ is denoted $v_1 * \cdots * v_n$ and is called a *decomposable tensor*.

The inner product on V induces an inner product on $\bigotimes^n V$ the restriction to $V_\chi(G)$ of which satisfies

$$(u_1 * \cdots * u_n, v_1 * \cdots * v_n) = \frac{\chi(e)}{|G|} d_\chi^G(A) \tag{5}$$

where

$$A = [(u_i, v_j)], \tag{6}$$

and

$$d_\chi^G(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}. \tag{7}$$

With respect to this inner product,

$$\bigotimes^n V = V_{\chi_1}(G) \oplus \cdots \oplus V_{\chi_k}(G) \tag{8}$$

is an orthogonal direct sum.

As in [4] let $\Gamma_{n,m}$ be the set of all sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $1 \leq \alpha_i \leq m$. Define an equivalence relation $\sim$ on $\Gamma_{n,m}$ by setting $\alpha \sim \beta$ if there exists $\theta \in G$ such that $\alpha = \beta\theta = (\beta_{\theta(1)}, \ldots, \beta_{\theta(n)})$. Let Δ be a system of distinct representatives of the equivalence classes.

Let G_α be the stabilizer subgroup of α, i.e.,

$$G_\alpha = \{\sigma \in G : \alpha\sigma = \alpha\}. \tag{9}$$

Define

$$\overline{\Delta} = \left\{ \alpha \in \Delta : \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0 \right\}. \tag{10}$$

Let $\{e_1, \ldots, e_m\}$ be a basis of V. Denote by $e_\alpha^\otimes$ and e_α^* the tensors $e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(n)}$ and $e_{\alpha(1)} * \cdots * e_{\alpha(n)}$, respectively. Taking the norm of e_α^*, with respect to the induced inner product, one easily obtains the condition $e_\alpha^* \neq 0$ if and only if $\alpha \in \overline{\Delta}$. In particular, if $\{e_1, \ldots, e_m\}$ is an orthonormal basis of V, we have

$$(e_\alpha^*, e_\beta^*) = \begin{cases} 0 & \text{if } \alpha \not\sim \beta \\ \dfrac{\chi(e)}{|G|} \displaystyle\sum_{\sigma \in G_\beta} \chi(\sigma\theta) & \text{if } \alpha = \beta\theta \text{ for some } \theta \in G \end{cases} \tag{11}$$

For $\gamma \in \overline{\Delta}$, let $V_\gamma = \langle e_{\gamma\sigma}^\otimes : \sigma \in G \rangle$ and $V_\gamma^* = T(G,\chi)(V_\gamma) = \langle e_{\gamma\sigma}^* : \sigma \in G \rangle$. It follows that

$$V_\chi(G) = \bigoplus_{\gamma \in \overline{\Delta}} V_\gamma^* \tag{12}$$

is an orthogonal direct sum. In [1] Freese proved that

$$\dim V_\gamma^* = \frac{\chi(e)}{|G_\gamma|} \sum_{\sigma \in G_\gamma} \chi(\sigma). \tag{13}$$

(If χ is of degree one, then $\dim V_\gamma^* = 1$ for all $\gamma \in \overline{\Delta}$.) So from (12)

$$\dim V_\chi(G) = \sum_{\gamma \in \overline{\Delta}} \frac{\chi(e)}{|G_\gamma|} \left(\sum_{\sigma \in G_\gamma} \chi(\sigma) \right). \tag{14}$$

We have another expression for $\dim V_\chi(G)$ [4]:

$$\dim V_\chi(G) = \frac{\chi(e)}{|G|} \sum_{\sigma \in G} \chi(\sigma) m^{c(\sigma)} \tag{15}$$

where $c(\sigma)$ denotes the number of cycles, including cycles of length one, in the disjoint cycle factorization of σ.

In the subsequent discussion we assume that $\{e_1, \ldots, e_m\}$ is a given orthonormal basis of V.

Our first goal is the investigation of the existence of a subset S of $\Gamma_{n,m}$ for which $\{e_\gamma^* : \gamma \in S\}$ is an orthogonal basis of $V_\chi(G)$. The study is motivated by the work of Wang and Gong [6]. The second goal is to figure out the dimension of the corresponding symmetry classes of tensors.

In section 2 we study the case when G is a cyclic group. As an application, we give a proof of Fermat's "little" theorem.

In section 3 we study the case when G is a dihedral group.

In section 4 we turn our attention to the case that G is the alternating group A_4 or the full symmetric group S_4. Some open problems will be mentioned.

2. THE CYCLIC GROUPS

The subgroup C_n of $S_n (n \geq 2)$ generated by the element

$$r = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}$$

is the *cyclic group* of degree n. Since it is an abelian group, the irreducible representations of C_n are of degree 1. The n irreducible characters are given by (see [5], p. 35)

$$\lambda_h(r^k) = e^{2\pi i h k / n}, \qquad h = 0, \ldots, n-1. \tag{16}$$

Since λ_h is of degree 1, each V_γ^* has dimension 1 and hence $V_{\lambda_h}(G)$ has an orthogonal basis $\{e_\gamma^* : \gamma \in \overline{\Delta}\}$ by (11). The following theorem deals with $\dim V_{\lambda_h}(G)$, i.e., the number of elements in $\overline{\Delta}$.

THEOREM 2.1 *Let $G = C_n$ and $m \equiv \dim V$. Then*

$$\dim V_{\lambda_h}(G) = \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i h k / n} m^{\gcd(n,k)}, \qquad h = 0, \ldots, n-1$$

where $\gcd(n,k)$ is the greatest common divisor of n and k and $\gcd(n,0) \equiv n$.

Proof This follows directly from formula (15). The only thing we have to do is to compute $c(\sigma) \equiv$ the number of cycles in the cycle decomposition of σ (cycles of length 1 are also included). Notice that $\langle r^k \rangle$ has as many elements as the smallest power of r^k which gives the identity e. So the number of elements of $\langle r^k \rangle$ is

$n/\gcd(n,k)$. Hence we have

$$c(r^k) = \gcd(n,k) \tag{17}$$

and the result follows immediately. ■

We then have the following corollaries.

COROLLARY 2.2 *Let m be an integer greater than 1. For $h = 0,1,\ldots,n-1$,*

$$\frac{1}{n}\sum_{k=0}^{n-1}\cos\frac{2\pi hk}{n}m^{\gcd(n,k)}$$

is a positive integer and

$$\frac{1}{n}\sum_{k=0}^{n-1}\sin\frac{2\pi hk}{n}m^{\gcd(n,k)} = 0.$$

Proof If $\gamma = (1,2,\ldots,2)$, then $G_\gamma = \{e\}$ and hence $\dim V_\gamma^* = 1$ by (13). So by (14) $\dim V_{\lambda_h}(G) \geq \dim V_\gamma^* = 1$. Separating the expression of Theorem 2.1 into real and imaginary parts, we have the desired result. ■

COROLLARY 2.3 (Fermat) *If n is a prime number, then $m^n \equiv m \bmod n$ for any integer m.*

Proof Since $(-m)^n \equiv -m \bmod n$ we may assume m is nonnegative. Also, the statement is clearly true if m equals 0 or 1 so that we may assume $m > 1$. By Corollary 2.2, when $h = 0$, the number

$$\frac{1}{n}\sum_{k=0}^{n-1}m^{\gcd(n,k)}$$

is an integer. If n is prime, then $\gcd(n,k) = 1$ for $k = 1,\ldots,n-1$ and $\gcd(n,0) \equiv n$ and hence

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1}m^{\gcd(n,k)} &= \frac{1}{n}(m^n + m + \cdots + m)\\ &= \frac{1}{n}((m^n - m) + nm).\end{aligned}$$

Hence $(m^n - m)$ is divisible by n. ■

3. THE DIHEDRAL GROUPS

The subgroup D_n of $S_n (n \geq 3)$ generated by the elements

$$r = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & n & n-1 & \cdots & 3 & 2 \end{pmatrix}$$

is the *dihedral group of degree n*. The generators r and s satisfy

$$r^n = e = s^2 \qquad \text{and} \qquad srs = r^{-1}$$

(see [2], p. 50). In particular, $D_n = \{r^k, sr^k : 0 \leq k < n\}$.

For each integer h with $0 < h < n/2$, D_n has an irreducible character χ_h of degree 2 given by

$$\chi_h(r^k) = 2\cos\frac{2\pi hk}{n}, \qquad \chi_h(sr^k) = 0 \tag{18}$$

(see [5], p. 37). The other characters of D_n are of degree 1. The character χ_h is induced from the character λ_h of $C_n = \langle r \rangle$ given in (16).

THEOREM 3.1 *Let $G = D_n (n \geq 3)$, let $\chi = \chi_h (0 < h < n/2)$ and assume $m \equiv \dim V \geq 2$. There exists a subset S of $\Gamma_{n,m}$ for which $\{e^*_\gamma : \gamma \in S\}$ is an orthogonal basis of $V_\chi(G)$ if and only if $n \equiv 0 \bmod 4h_2$ where $h = h_2 h_{2'}$ with h_2 a power of 2 and $h_{2'}$ odd.*

Proof Assume $V_\chi(G)$ has an orthogonal basis $\{e^*_\gamma : \gamma \in S\}$ consisting of decomposable symmetrized tensors. It will first be shown that $\chi(\sigma) = 0$ for some $\sigma \in C_n$.

Let $\gamma = (1,2,2,\ldots,2)$ and note that $s \in G_\gamma$ and $G_\gamma \cap C_n = \{e\}$. Moreover, if $sr^k \in G_\gamma$ with $0 \leq k < n$, then $r^k = ssr^k \in G_\gamma \cap C_n$, so that $k = 0$. Hence $G_\gamma = \{e, s\}$. It is easy to check that

$$\tau^{-1}G_\gamma\mu = \begin{cases} \{r^{k-j}, sr^{j+k}\}, & \text{if } \tau = r^j, \quad \mu = r^k, \\ \{sr^{j+k}, r^{k-j}\}, & \text{if } \tau = r^j, \quad \mu = sr^k, \\ \{r^{k-j}, sr^{j+k}\}, & \text{if } \tau = sr^j, \quad \mu = sr^k. \end{cases}$$

In particular, $|\tau^{-1}G_\gamma\mu \cap C_n| = 1$ for all $\tau, \mu \in G$. We have from (11) that

$$\begin{aligned} (e^*_{\gamma\mu}, e^*_{\gamma\tau}) &= \frac{\chi(e)}{|G|} \sum_{\sigma \in G_{\gamma\tau}} \chi(\sigma\tau^{-1}\mu) \\ &= \frac{\chi(e)}{|G|} \sum_{\sigma \in \tau^{-1}G_\gamma\tau} \chi(\sigma\tau^{-1}\mu) \\ &= \frac{\chi(e)}{|G|} \sum_{\sigma \in \tau^{-1}G_\gamma\mu} \chi(\sigma) \\ &= \frac{\chi(e)}{|G|} \sum_{\sigma \in \tau^{-1}G_\gamma\mu \cap C_n} \chi(\sigma) \end{aligned} \tag{19}$$

If χ never vanished on C_n, then it would follow that $(e^*_{\gamma\tau}, e^*_{\gamma\mu}) \neq 0$ for each $\tau, \mu \in G$ and, since $\dim V^*_\gamma = 2$ by (13), this would contradict that V^*_γ has an orthogonal basis of decomposable symmetrized tensors.

Therefore, $2\cos(2\pi hk/n) = \chi(r^k) = 0$ for some k. In other words, $2\pi hk/n = (2\ell + 1)\pi/2$ for some integer ℓ. This implies $4hk = (2\ell + 1)n$ so that $4h_2$ must divide n.

Conversely, assume $n \equiv 0 \bmod 4h_2$. It will first be shown that if $\gamma \in \overline{\Delta}$, then G_γ is of the form H or HT where H is a subgroup of $\langle r^{n'} \rangle$ with $n' = n/\gcd(n,h)$ and $T = \langle t \rangle$ with $t^2 = e$ and $t \notin C_n$.

Set $H = G_\gamma \cap C_n$ which is obviously a subgroup of C_n. Recall that χ equals the induced character λ^G where $\lambda = \lambda_h$. Denoting by χ_H and λ_H the restrictions to H of χ and λ, respectively, Mackey's subgroup theorem (see [5], p. 58) implies

$$\chi_H = (\lambda^G)_H = \lambda_H + (\lambda_H)^s$$

where $(\lambda_H)^s$ is the character of H defined by

$$\begin{aligned}(\lambda_H)^s(x) &= \lambda_H(s^{-1}xs)\\ &= \lambda_H(x^{-1})\\ &= \lambda_H(x)^{-1}, \qquad x \in H.\end{aligned} \tag{20}$$

Note that from (20), $\lambda_H = 1$ if and only if $(\lambda_H)^s = 1$. Now

$$\begin{aligned}(\lambda_H + (\lambda_H)^s, 1)_H &= (\chi_H, 1)_H\\ &= \frac{1}{|H|}\sum_{\sigma \in H} \chi(\sigma)\\ &= \frac{1}{|H|}\sum_{\sigma \in G_\gamma} \chi(\sigma)\\ &\neq 0,\end{aligned}$$

since $\gamma \in \overline{\Delta}$, so it follows that $\lambda_H = 1$.

If $r^k \in H$, then $\lambda(r^k) = 1$, so $2\pi hk/n = 2\pi\ell$ for some integer ℓ. Therefore, $h'k = \ell n'$ where $h' = h/\gcd(n,h)$. Since ℓ/h' is an integer, $r^k \in \langle r^{n'} \rangle$. Consequently, H is a subgroup of $\langle r^{n'} \rangle$.

Suppose $G_\gamma \neq H$. Then G_γ contains some $t \in G \backslash C_n$. In particular $t^2 = e$, so $T = \{e, t\}$ is a subgroup of G_γ. Since H is a normal subgroup of G_γ, HT is a subgroup of G_γ. Also $|HT| = 2|H|$. Now $C_n G_\gamma = G$, so by an isomorphism theorem $G_\gamma/H \cong G/C_n \cong \mathbb{Z}/2\mathbb{Z}$. In particular, $|G_\gamma| = 2|H|$, so that $G_\gamma = HT$, as desired.

Let $\gamma \in \overline{\Delta}$. It will be shown that V_γ^* has an orthogonal basis $\{e_{\gamma\sigma}^* : \sigma \in K\}$ where

$$K = \begin{cases} \{e, r^{n'/4}, s, sr^{n'/4}\} & \text{if } G_\gamma = H,\\ \{e, r^{n'/4}\} & \text{if } G_\gamma = HT.\end{cases} \tag{21}$$

Recall that $\chi_H = \lambda_H + (\lambda_H)^s = 2 \cdot 1_H$ so that χ is identically 2 on H. Therefore, it follows from (13) that if $G_\gamma = H$, then $\dim V_\gamma^* = 4$ and if $G_\gamma = HT = H \dot\cup Ht$, then $\dim V_\gamma^* = 2$. Consequently, it remains to be shown that $\{e_{\gamma\sigma}^* : \sigma \in S\}$ consists of mutually orthogonal vectors, and for this it is enough, by (19) to show that $\chi(\tau^{-1}G_\gamma\mu) = \{0\}$ for each pair $\tau, \mu \in S$ with $\tau \neq \mu$.

First consider the case $G_\gamma = H$. If $\tau \in C_n$ and $\mu \notin C_n$, then $\tau^{-1}G_\gamma\mu \cap C_n = \phi$ so that $\chi(\tau^{-1}G_\gamma\mu) = \{0\}$. If $\tau = e$ and $\mu = r^{n'/4}$ or $\tau = s$ and $\mu = sr^{n'/4}$, then

$\tau^{-1}\langle r^{n'}\rangle\mu = \{r^{n'(1+4k\epsilon)/4} : k \in \mathbb{Z}\}$ where ϵ is 1 for the first case and -1 for the second case. Now

$$\chi(r^{n'(1+4k\epsilon)/4}) = 2\cos\frac{2\pi h n'(1+4k\epsilon)/4}{n} = 2\cos\frac{h'(1+4k\epsilon)\pi}{2} = 0,$$

since h' is odd. Therefore, $\chi(\tau^{-1}G_\gamma\mu) = \{0\}$ as G_γ is contained in $\langle r^{n'}\rangle$.

Finally assume $G_\gamma = HT = H\dot{\cup}Ht$ so that $\tau^{-1}G_\gamma\mu = \tau^{-1}H\mu\dot{\cup}\tau^{-1}Ht\mu$. If $\tau = e$ and $\mu = r^{n'/4}$, then $\chi(\tau^{-1}H\mu) = \{0\}$ by the preceding paragraph and $\chi(\tau^{-1}Ht\mu) = \{0\}$ since $\tau^{-1}Ht\mu \cap C = \phi$. Hence $\chi(\tau^{-1}G_\gamma\mu) = \{0\}$, as desired. ■

COROLLARY 3.2 [6] *There exists a subset S of $\Gamma_{4,m}$ for which $\{e^*_\gamma : \gamma \in S\}$ is an orthogonal basis of $V_{\chi_1}(D_4)$.*

Remark 1 In [6] Corollary 3.2 was obtained by the explicit construction of an orthogonal basis of $V_{\chi_1}(D_4)$. The purpose was to point out a false statement in [3].

Remark 2 The second half of the proof of Theorem 3.1 does not make use of the given embedding of the dihedral group in the full symmetric group. In fact, it shows that if G is any subgroup of S_n isomorphic to D_ℓ for some $\ell \geq 3$, then for each $0 < h < \ell/2$ satisfying $\ell \equiv 0 \bmod 4h_2$, there exists a subset S of $\Gamma_{n,m}$ for which $\{e^*_\gamma : \gamma \in S\}$ is an orthogonal basis of $V_{\chi_h}(G)$.

We will say that $\otimes^n V$ has an *orthogonal basis of decomposable symmetrized tensors* if for each irreducible character χ of G, there exists a subset S_χ of $\Gamma_{n,m}$ for which $\{e^*_\gamma : \gamma \in S_\chi\}$ is an orthogonal basis of $V_\chi(G)$.

COROLLARY 3.3 *Let $G = D_n$ and assume that* $\dim V \geq 2$. *Then $\otimes^n V$ has an orthogonal basis of decomposable symmetrized tensors if and only if n is a power of* 2.

Proof Keeping the earlier notation, let n_2 denote the largest power of 2 dividing n. Assume that $n_2 < n$. Then $0 < n_2 < n/2$ and $n \not\equiv 0 \bmod 4n_2$. Therefore, if $\chi = \chi_{n_2}$, then Theorem 3.1 implies that there exists no subset S_χ of $\Gamma_{n,m}$ for which $\{e^*_\gamma : \gamma \in S_\chi\}$ is an orthogonal basis for $V_\chi(G)$.

Conversely, assume n is a power of 2. If $0 < h < n/2$, then $h_2 \leq n/4$ so that $n \equiv 0 \bmod 4h_2$. Theorem 3.1 now implies that $\otimes^n V$ has an orthogonal basis of decomposable symmetrized tensors (recalling that the irreducible characters of G not of the form χ_h are all of degree one). ■

If n is even, there are 4 irreducible characters of degree 1, given by the following table:

	r^k	sr^k
ψ_1	1	1
ψ_2	1	-1
ψ_3	$(-1)^k$	$(-1)^k$
ψ_4	$(-1)^k$	$(-1)^{k+1}$

If n is odd, then ψ_1 and ψ_2 are the only irreducible characters of degree 1.

THEOREM 3.4 *Let $G = D_n (n \geq 3)$. Assume $m \equiv \dim V \geq 2$. Then*

(a)

$$\dim V_{\psi_1}(G) = \begin{cases} \frac{1}{4}m^{n/2}(m+1) + \frac{1}{2n}\sum_{k=0}^{n-1} m^{\gcd(n,k)} & \text{if } n \text{ is even} \\ \frac{1}{2}m^{(n+1)/2} + \frac{1}{2n}\sum_{k=0}^{n-1} m^{\gcd(n,k)} & \text{if } n \text{ is odd} \end{cases}$$

(b)

$$\dim V_{\psi_2}(G) = \begin{cases} -\frac{1}{4}m^{n/2}(m+1) + \frac{1}{2n}\sum_{k=0}^{n-1} m^{\gcd(n,k)} & \text{if } n \text{ is even} \\ -\frac{1}{2}m^{(n+1)/2} + \frac{1}{2n}\sum_{k=0}^{n-1} m^{\gcd(n,k)} & \text{if } n \text{ is odd} \end{cases}$$

(c)

$$\dim V_{\psi_3}(G) = \frac{1}{4}m^{n/2}(m-1) + \frac{1}{2n}\sum_{k=0}^{n-1}(-1)^k m^{\gcd(n,k)} \qquad \text{if } n \text{ is even}$$

(d)

$$\dim V_{\psi_4}(G) = \frac{1}{4}m^{n/2}(1-m) + \frac{1}{2n}\sum_{k=0}^{n-1}(-1)^k m^{\gcd(n,k)} \qquad \text{if } n \text{ is even}$$

(e)

$$\dim V_{\chi_h}(G) = \frac{2}{n}\sum_{k=0}^{n-1} \cos\frac{2\pi hk}{n} m^{\gcd(k,n)}, \qquad 0 < h < \frac{n}{2}.$$

Proof Notice that

$$(sr^k)^2 = (sr^k s)r^k = r^{-k}r^k = e.$$

This implies that the cycles of sr^k are of length either 1 or 2. Let (ℓ) be a cycle of length 1 in sr^k, i.e., $sr^k(\ell) = \ell$ and hence $r^k(\ell) = s(\ell)$. So we have

$$\begin{cases} k + \ell \equiv n + 2 - \ell \bmod n & \text{if } \ell \neq 1 \\ k + \ell = 1 & \text{if } \ell = 1 \end{cases}. \tag{22}$$

If n is even, then (22) has a solution ℓ if and only if k is even. In this case, $\ell = 1$ or $\ell = (n+2)/2$ if $k = 0$ and $\ell = (n+2-k)/2$ or $\ell = (2n+2-k)/2$ if $k > 0$. If n is odd, then (22) always has a solution $\ell = (n+2-k)/2$ if k is odd or $\ell = (2n +$

$2-k)/2$ if $k \neq 0$ if even and $\ell = 1$ if $k = 0$. The other cycles are of length 2. Hence

$$c(sr^k) = \begin{cases} \frac{1}{2}(n+1+(-1)^k) & \text{if } n \text{ is even} \\ \frac{1}{2}(n+1) & \text{if } n \text{ is odd.} \end{cases}$$

Then we have the desired result by (15) and (17). ■

4. THE SYMMETRIC GROUP S_4 AND THE ALTERNATING GROUP A_4

The symmetric group S_4 is the group of all permutations of a set $\{a, b, c, d\}$ having four elements. There are 5 conjugacy classes:

$$\{e\},$$

$$\{(ab), (ac), (ad), (bc), (bd), (cd)\},$$

$$\{(ab)(cd), (ac)(bd), (ad)(bc)\},$$

$$\{(abc), (acb), (abd), (adb), (acd), (adc), (bcd), (bdc)\} \quad \text{and}$$

$$\{(abcd), (abdc), (acbd), (acdb), (adbc), (adcb)\}.$$

Let $x = (ab)(cd)$, $y = (ac)(bd)$, $z = (ad)(bc)$, and let L be the group of permutations that leave d fixed. We have the following character table (see [5], p. 43).

	e	(ab)	$(ab)(cd)$	(abc)	$(abcd)$
χ_0	1	1	1	1	1
ϵ	1	-1	1	1	-1
θ	2	0	2	-1	0
ψ	3	1	-1	0	-1
$\epsilon\psi$	3	-1	-1	0	1

EXAMPLE 4.1 There is no subset S of $\Gamma_{4,m}$ for which $\{e_\gamma^* : \gamma \in S\}$ is an orthogonal basis of $V_\psi(S_4)$ if $m \equiv \dim V \geq 2$.

Proof It is sufficient to show that there is a $\gamma \in \overline{\Delta}$ such that V_γ^* does not have an orthogonal basis among $\{e_{\gamma\sigma}^* : \sigma \in S_4\}$. Let $\gamma = (1, 1, 1, 2)$. Then $G_\gamma = L$ and by (13),

$$\dim V_\gamma^* = \frac{3}{6} \sum_{\sigma \in G_\gamma} \psi(\sigma) = 3.$$

Now $S_4 = LH$, so $\{e_{\gamma\sigma}^* : \sigma \in S_4\} = \{e_{\gamma\sigma}^* : \sigma \in H\}$. By using formula (19), we have the following table for the inner products $(e_{\gamma\mu}^*, e_{\gamma\tau}^*)$.

$\mu\backslash\tau$	e	x	y	z
e	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
x	$-\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
y	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$
z	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$

So it is impossible to find three orthogonal vectors among these four vectors. ■

The alternating group A_4 is the group of even permutations of a set $\{a,b,c,d\}$ having 4 elements. There are 4 conjugacy classes in A_4:

$$\{e\},$$
$$\{x,y,z\},$$
$$\{t,tx,ty,tz\}, \qquad \text{and}$$
$$\{t^2,t^2x,t^2y,t^2z\},$$

where

$$t=(abc), \qquad x=(ab)(cd), \qquad y=(ac)(bd), \qquad z=(ad)(bc).$$

We have

$$txt^{-1}=z, \qquad tzt^{-1}=y, \qquad tyt^{-1}=x$$

and the following character table (see [5], p. 42)

	e	x	t	t^2
χ_0	1	1	1	1
χ_1	1	1	ω	ω^2
χ_2	1	1	ω^2	ω
ψ	3	-1	0	0

with $\omega = e^{2\pi i/3}$.

EXAMPLE 4.2 There is no subset S of $\Gamma_{4,m}$ for which $\{e_\gamma^* : \gamma \in S\}$ is an orthogonal basis of $V_\psi(A_4)$ if $m \equiv \dim V \geq 2$.

Proof It is sufficient to find a $\gamma \in \Gamma_{4,m}$ such that V_γ^* does not have an orthogonal basis among $\{e_{\gamma\sigma}^* : \sigma \in A_4\}$. Let $\gamma = (1,1,1,2)$. Then $G_\gamma = \{e,t,t^2\}$ and

$$\dim V_\gamma^* = \frac{3}{3}\sum_{\sigma\in G_\gamma}\psi(\sigma) = 3.$$

Moreover $A_4 = G_\gamma H$ where $H = \{e,x,y,z\}$, so $\{e_{\gamma\sigma}^* : \sigma \in A_4\} = \{e_{\gamma\sigma}^* : \sigma \in H\}$.

By using formula (19), we have the following table for the inner products $(e^*_{\gamma\mu}, e^*_{\gamma\tau})$.

$\mu\backslash\tau$	e	x	y	z
e	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
x	$-\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
y	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$
z	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$

So it is impossible to find three orthogonal vectors among these four vectors. ■

In view of the above examples, we conjecture that for $G = S_n$ or A_n, $n > 4$, there is a character χ of G such that no subset S of $\Gamma_{n,m}$ exists for which $\{e^*_\gamma : \gamma \in S\}$ is an orthogonal basis of $V_\chi(G)$. (Notice that $S_3 = D_3$ and $S_2 = D_2 = C_2$.)

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